Worcester County Mathematics League

Varsity Meet 3
January 25, 2017

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 25, 2017

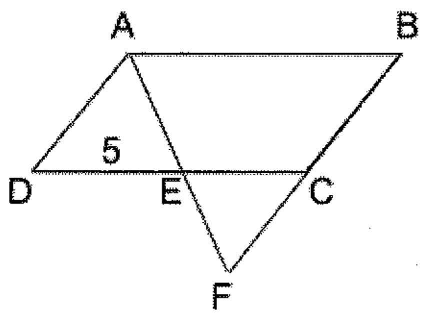
Round 1: Similarity and Pythagorean Theorem

All answers must be in simplest exact form in the answer section

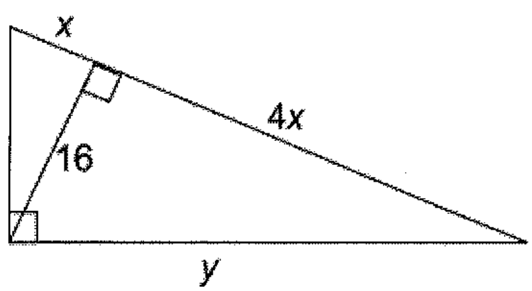
NO CALCULATOR ALLOWED

1. Find the distance between (-4, 2) and (11, 1).

2. If ABCD is a parallelogram and the ratio of CF to BC is 2 to 3, find EC.



3. Find y in the figure:



ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 25, 2017

Round 2: Algebra 1

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $x+y=10$ and $x=\frac{7}{y}$, find x^2+y^2.

2. Solve: $\sqrt[3]{2x-1}=\sqrt[6]{x+1}$

3. Solve: $|x-3| = -|x|+7$

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 25, 2017

Round 3: Functions

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $f(x) = 5x^2 + 2x - 3$, compute $f(2x-1)$ and express it as an expanded polynomial.

2. Let a, b and c be non-zero real numbers and suppose that $f(x) = x^2 + bx + c$. If $f(a+1) = f(a-1)$, compute $\frac{a}{b}$.

3. Suppose that $f(2) = 9, f(3) = 8, f(4) = 6$. If $f^{-1}(x) = 2g(x)$ and $g^{-1}(x) = 3h(x)$, find $h(1)$.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

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Round 4: Combinatorics

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. How many different diagonals can be drawn within a regular octagon?

2. At Pizza Palace, you can order a small, medium or large pizza and can choose toppings of pepperoni, mushrooms, bacon, olives and anchovies. How many different ways can you order a pizza with at least one topping?

3. How many different amounts of money can be made from a collection of coins consisting of 3 pennies, 2 nickels, 1 dime and 1 quarter, where at least one coin is used?

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 25, 2017

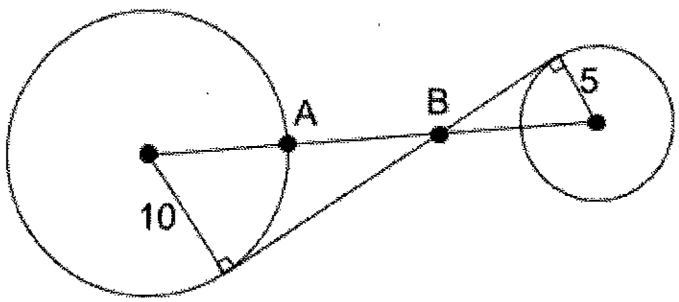
Round 5: Analytic Geometry

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Determine the area of the region enclosed by the curve $y^2+x^2=16$ that is below the line $y= -x+4$. Express your answer in terms of π.

2. Find the length of segment AB in the figure if the distance between the centers of the circles is 20 inches:



3. Find one possible y-intercept for a line $y=\frac{3}{4}x+b$ which is a distance of 6 units from the point (2, 11).

ANSWERS

(1 pt.) 1. ______________________ square units

(2 pts.) 2. ______________________ inches

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 25, 2017

Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 points each)

APPROVED CALCULATORS ALLOWED

1. If $f(x)$ and $g(x)$ are linear functions such that $f(0) = g(0) = 0$. If $f(g(x)) = 8x$, then evaluate $g(f(10))$.

2. A committee of 5 students is to be chosen at random from 4 juniors and 7 seniors. If the committee is chosen randomly and each student has an equal chance to be chosen, what is the probability that the committee will contain exactly 3 seniors and 2 juniors?

3. If $f(x) = \frac{\sqrt{x}}{2+\sqrt{x}}$ and $g(x) = \left(\frac{2x}{1-x}\right)^2$, then compute $g(f(x))$ and simplify.

4. Bob and Jill meet 2 other married couples at a restaurant. No person shakes hand with his or her spouse. Excluding Bob, each other person shakes hands with a different number of people (possibly with no one) and does not shake hands with the same person twice. With how many people did Bob shake hands?

5. Find the x coordinate of the focus of the conic section given by $y^2 + 4y - 6x + 22 = 0$.

6. Solve and graph on the real number line: $x(x+2)(x-3) > 0$

7. Let x be the largest integer such that 2^x divides 100! evenly. Compute x.

8. Two positive numbers a and b have the property that their sum is equal to their product. If $a - b = 3$, then determine the value of $\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}}$.

9. The lengths of the legs of a right triangle are 3 and 4. Two congruent circles, externally tangent to each other, are drawn inside the triangle such that each circle is tangent to the hypotenuse and one of the legs. Determine the distance between the circles' centers.

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 25th, 2017 ANSWER KEY

Round 1:

1. $\sqrt{226}$ (Bartlett)
2. $3\frac{1}{3}$ or $\frac{10}{3}$ or $3.\overline{3}$ (Notre Dame Academy)
3. $16\sqrt{5}$ (Bancroft)

Round 2:

1. 86 (Doherty)
2. $\frac{5}{4}$ or $1\frac{1}{4}$ or 1.25 (Tantasqua)
3. 5 and -2 (*Both required)* (Shrewsbury)

Round 3:

1. $20x^2 - 16x$ (Douglas)
2. $-\frac{1}{2}$ or -0.5 (Quaboag)
3. 3 (Shrewsbury)

Round 4:

1. 20 (Southbridge)
2. 93 (Shrewsbury)
3. 39 (St. John's)

Round 5:

1. $12\pi + 8$ (Doherty)
2. $\frac{10}{3}$ or $3\frac{1}{3}$ or $3.\overline{3}$ (Auburn)
3. 2 or 17 (*Note: Only one of the answers is required)* (Leicester)

WORCESTER COUNTY MATHEMATICS LEAGUE

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Team Round Answer Sheet

1. ____________________

2. ____________________

3. ____________________

4. ____________________

5. ____________________

6. 

7. ____________________

8. ____________________

9. ____________________ units

TEAM Round

1. 80 (Bromfield)

2. $\frac{5}{11}$ or $0.\overline{45}$ (St. Peter-Marian)

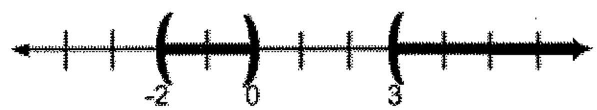
3. x (Hudson)

4. 2 (Northbridge)

5. $\frac{9}{2}$ or $4\frac{1}{2}$ or 4.5 (Hudson)

6. (Bromfield)

--- OR ---



7. 97 (Leicester)

8. $\frac{13+2\sqrt{13}}{13}$ or $1+\frac{2\sqrt{13}}{13}$ or 1.555 (*Note: Not repeating)* (AMSA)

9. $\frac{10}{7}$ or $1\frac{3}{7}$ or 1.429 (Tahanto)

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 25, 2017 - SOLUTIONS

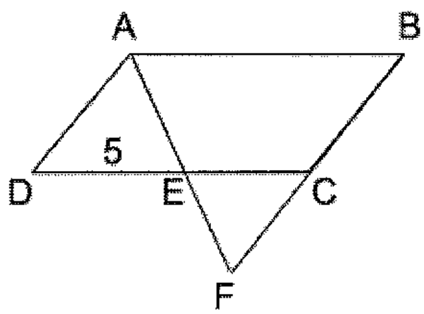
Round 1: Similarity and Pythagorean Theorem

1. Find the distance between (-4, 2) and (11, 1).

Solution: To find the distance d between the two points, we can use the distance formula, an application of the Pythagorean Theorem. We have that

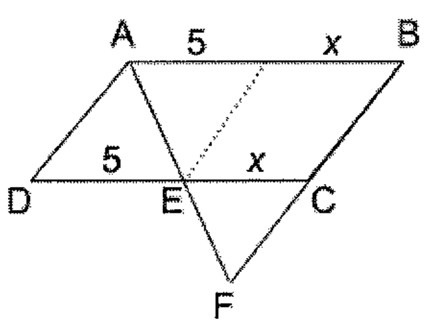
$$d=\sqrt{(11-(-4))^2+(1-2)^2}$$
$$d=\sqrt{15^2+(-1)^2}$$
$$d=\sqrt{226}$$

2. If ABCD is a parallelogram and the ratio of CF to BC is 2 to 3, find EC.



Solution 1: Since the ratio of CF to BC is 2 to 3, we have that the ratio of CF to FB is 2 to 5.

Let x be the length EC. Now draw in the dotted line shown below which is parallel to segment BC. This shows us that AB = $5+x$.



Now we use the fact that triangle CEF is similar to triangle BAF to see that

$$\frac{2}{5} = \frac{x}{5+x}$$

$$2(5+x) = 5x$$

$$10 + 2x = 5x$$

$$3x = 10$$

$$x = \frac{10}{3}$$

Solution 2: Let b be the length of segment EC and let $2x$ be the length of segment CF. By the ratio information provides, we know that the length of segment BC is $3x$. By congruent opposite sides of a parallelogram, we therefore know that the length of segment AD is also $3x$.

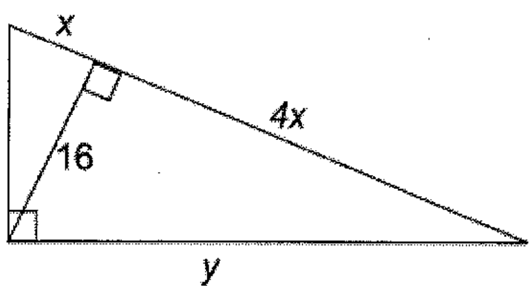
Now note that triangle ADE is similar to triangle FCE, which gives us that

$$\frac{3x}{5} = \frac{2x}{b}$$

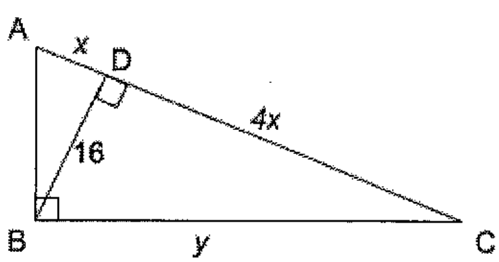
$$b(3x) = 5(2x)$$

$$b = \frac{2x}{3x} \times 5 = \frac{2 \times 5}{3} = \frac{10}{3}.$$

3. Find y in the figure:



Solution 1: Begin by labeling the vertices in the figure. Notice that triangle BCD is similar to triangle ADB.



By similarity, we have that

$$\frac{4x}{16} = \frac{16}{x}$$

$$4x^2 = 16^2 \rightarrow 2x = 16 \rightarrow x = 8.$$

Now that we know the value of x, the Pythagorean Theorem gives us that

$$(4x)^2 + (16)^2 = y^2$$
$$32^2 + 16^2 = y^2$$
$$2^2 \times 16^2 + 16^2 = y^2$$
$$(4+1) \times 16^2 = y^2$$
$$y = 16\sqrt{5}$$

Solution 2: Begin by solving for the unknown hypotenuse of the small triangle. By the Pythagorean Theorem, we know that it is equal to $\sqrt{16^2 + x^2}$.

Now in the triangle with sides 16, $4x$, and y, we can use the Pythagorean Theorem again to see that $y^2 = 16^2 + 16x^2$.

Now in the triangle with sides $5x$, y, and $\sqrt{16^2 + x^2}$, we have that

$$(5x)^2 = y^2 + (16^2 + x^2)$$
$$25x^2 = (16^2 + 16x^2) + 16^2 + x^2$$
$$8x^2 = 2 \times 16^2 = 2 \times (8^2 \times 2^2)$$
$$x^2 = \tfrac{1}{8} \times 2^3 \times 8^2 = 8^2 \quad \Rightarrow \quad x = 8.$$

Since $y^2 = 16^2 + 16x^2$, we now have that

$$y = \sqrt{16^2 + 16 \times 8^2}$$
$$y = \sqrt{16^2 + 16 \times (2^2 \times 4^2)}$$
$$y = \sqrt{16^2 + 16^2 \times 2^2}$$
$$y = 16\sqrt{1 + 2^2} = 16\sqrt{5}.$$

Round 2: Algebra 1

1. If $x + y = 10$ and $x = \frac{7}{y}$, find $x^2 + y^2$.

Solution: From the second piece of given information, we have that $xy = 7$.

Now notice that we have

$$x + y = 10$$
$$(x+y)^2 = 10^2$$
$$x^2 + 2xy + y^2 = 100$$
$$x^2 + 2(7) + y^2 = 100$$
$$x^2 + y^2 = 86$$

2. Solve: $\sqrt[3]{2x-1} = \sqrt[6]{x+1}$

Solution: Begin by raising both sides of the equation to the sixth power. This gives

$$(2x-1)^2 = x+1$$
$$4x^2 - 4x + 1 = x + 1$$
$$4x^2 - 5x = 0$$
$$x(4x-5) = 0$$

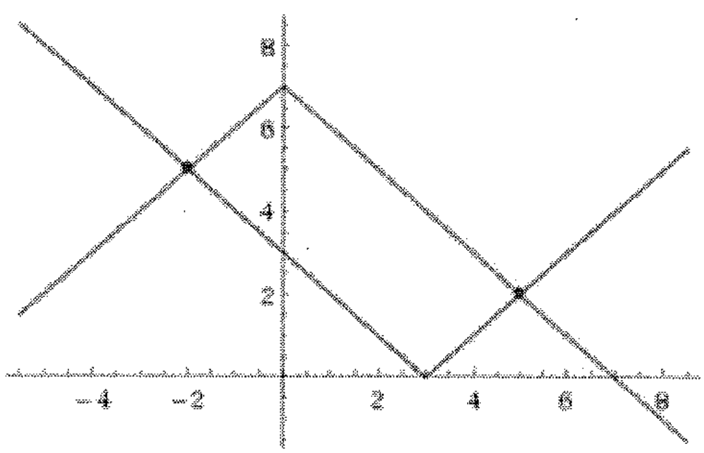
Therefore, our possible solutions are $x = 0$ and $x = \frac{5}{4}$. However, notice that the solution of $x = 0$ does not work since it would give that $-1 = 1$. Hence, we have that $x = \frac{5}{4}$.

3. Solve: $|x-3| = -|x| + 7$

Solution 1 (Algebraic): We have that

$$|x-3| = -|x| + 7$$
$$|x-3| + |x| = 7$$
$$(x-3) + (x) = 7 \quad \text{or} \quad (x-3) + (x) = -7$$
$$2x - 3 = 7 \quad \text{or} \quad 2x - 3 = -7$$
$$2x = 10 \quad \text{or} \quad 2x = -4$$
$$x = 5 \quad \text{or} \quad x = -2$$

Solution 2 (Graphing): We can graph both sides of this equation to quickly find the solution. We have



We see that the x values which satisfy the equation are 5 and -2.

Round 3: Functions

1. If $f(x) = 5x^2 + 2x - 3$, compute $f(2x-1)$ and express it as an expanded polynomial.

Solution: We have that

$$\begin{aligned} f(2x-1) &= 5(2x-1)^2 + 2(2x-1) - 3 \\ f(2x-1) &= 5(4x^2 - 4x + 1) + 4x - 2 - 3 \\ f(2x-1) &= 20x^2 - 20x + 5 + 4x - 5 \\ f(2x-1) &= 20x^2 - 16x \end{aligned}$$

2. Let a, b and c be non-zero real numbers and suppose that $f(x) = x^2 + bx + c$. If $f(a+1) = f(a-1)$, compute $\frac{a}{b}$.

Solution: We have that

$$\begin{aligned} f(a+1) &= f(a-1) \\ (a+1)^2 + b(a+1) + c &= (a-1)^2 + b(a-1) + c \\ (a^2 + 2a + 1) + ba + b &= (a^2 - 2a + 1) + ba - b \\ 4a = -2b \quad &\rightarrow \quad \frac{a}{b} = -\frac{1}{2} \end{aligned}$$

3. Suppose that $f(2) = 9$, $f(3) = 8$, $f(4) = 6$. If $f^{-1}(x) = 2g(x)$ and $g^{-1}(x) = 3h(x)$, find $h(1)$.

Solution 1: Start by noticing that

x	2	3	4
$f(x)$	9	8	6

This gives us that

x	9	8	6
$f^{-1}(x)$	2	3	4

We are given that $f^{-1}(x) = 2g(x)$, which allows us to determine that

x	9	8	6
$g(x) = \frac{1}{2}f^{-1}(x)$	1	1.5	2

Next we can compute that

x	1	1.5	2
$g^{-1}(x)$	9	8	6

Finally, we are given that $g^{-1}(x) = 3h(x)$, which allows us to determine that

x	1	1.5	2
$h(x) = \frac{1}{3}g^{-1}(x)$	3	$\frac{8}{3}$	2

So we have that $h(1) = 3$.

Solution 2: Begin by noting that $f^{-1}(9) = 2$. Since we are given that $f^{-1}(x) = 2g(x)$, we have that $g(x) = \frac{1}{2}f^{-1}(x)$. This means that

$$g(9) = \tfrac{1}{2}f^{-1}(9) = \tfrac{1}{2} \times 2 = 1 \quad \Rightarrow \quad g^{-1}(1) = 9.$$

Similarly, since we are given that $g^{-1}(x) = 3h(x)$, we have that $h(x) = \frac{1}{3}g^{-1}(x)$. This means

$$h(1) = \tfrac{1}{3}g^{-1}(1) = \tfrac{1}{3} \times 9 = 3.$$

Round 4: Combinatorics

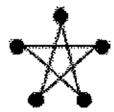
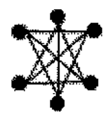
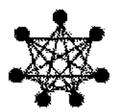
1. How many different diagonals can be drawn within a regular octagon?

Solution 1: There are eight vertices in an octagon. Since we cannot draw a diagonal from a vertex to either of its adjacent vertices, we have that we can draw exactly five diagonals from each vertex in the octagon.

However, we must be careful not to double count diagonals, as each diagonal is connected to two vertices. Therefore, the total number of diagonals is given by

$$\frac{8 \times 5}{2} = 20$$

Solution 2: First consider how many diagonals are contained in polygons with fewer sides. We have that

Triangle		0 diagonals
Square		2 diagonals (1 + 1)
Pentagon		5 diagonals (2 + 2 + 1)
Hexagon		9 diagonals (3 + 3 + 2 + 1)
Septagon		14 diagonals (4 + 4 + 3 + 2 + 1)
n-gon	???	$(n - 3) + (n - 3) + (n - 4) + (n - 5) + \ldots + (n - (n - 1))$
⇒ Octagon	???	(5) + (5) + (4) + (3) + (2) + (1) = 20

2. At Pizza Palace, you can order a small, medium or large pizza and can choose toppings of pepperoni, mushrooms, bacon, olives and anchovies. How many different ways can you order a pizza with at least one topping?

Solution: We have that there are three different sizes of pizza and that we can choose 1, 2, 3, 4 or 5 toppings total. The number of possible combinations is therefore given by

$$3 \times \left[\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right]$$

$$3 \times \left[\frac{5!}{1!\times 4!} + \frac{5!}{2!\times 3!} + \frac{5!}{3!\times 2!} + \frac{5!}{4!\times 1!} + \frac{5!}{5!\times 0!}\right]$$

$$3 \times [5 + 10 + 10 + 5 + 1] = 93$$

3. How many different amounts of money can be made from a collection of coins consisting of 3 pennies, 2 nickels, 1 dime and 1 quarter, where at least one coin is used?

Solution 1: Begin by noting that if we use at least one coin only from the set of pennies, there are three possible amounts of cents: 1, 2, 3

If we use at least one coin only from the set of two nickels, one dime and one quarter, we can get nine possible amounts of cents: 5, 10, 15, 20, 25, 30, 35, 40, 45. For each of these nine amounts, we can choose to add zero, one, two, or three pennies.

Therefore, in total we have $3+(9\times 4)=39$ different amounts of money that can be made.

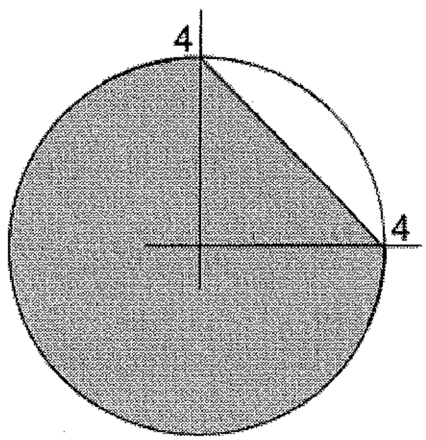
Solution 2: Begin by noting that there are 3 possible amounts of nickels, namely 0, 1 or 2. Similarly there are 2 possible amounts of dimes: 0 and 1. Combined, this gives six different possible combinations of nickels and dimes, but only five unique monetary values since 2 nickels and 0 dimes is equal to 0 nickels and 1 dime.

Since there are four possible amounts of pennies and two possible amounts of quarters, we have that there are a total of $4\times 2\times (6-1)=40$ possible monetary amounts. However, since we must use at least 1 coin, we must not count combination of 0 pennies, 0 nickels, 0 dimes and 0 quarters. Hence there are $40-1=39$ possible monetary amounts.

Round 5: Analytic Geometry

1. Determine the area of the region enclosed by the curve $y^2+x^2=16$ that is below the line $y= -x+4$. Express your answer in terms of π.

Solution: Begin by drawing the two given curves and shading in the desired area.

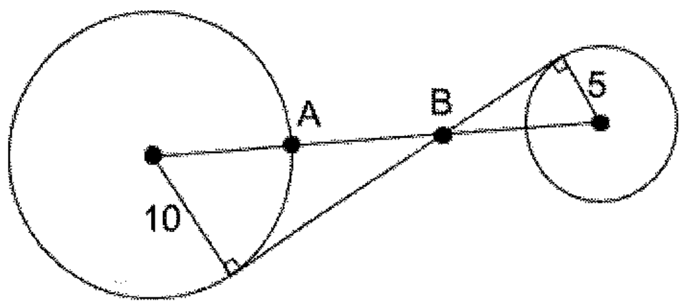


We have that the second given curve is a circle with radius 4 with its center located at the origin. Therefore, we know the area of the circle is 16π.

To compute by the desired area, we must subtract the area of region of the circle which is above the given line. Notice that this area is equal to ¼ of the area of the circle minus the area of the right triangle in the first quadrant.

Therefore, we have that the shaded region has an area of $16\pi-\left(\frac{1}{4}(16\pi)-\frac{1}{2}(4)(4)\right)=12\pi+8$ sq units.

2. Find the length of segment AB in the figure if the distance between the centers of the circles is 20 inches:



Solution 1: Begin by noticing that the two triangles in the figure are similar to each other. Let x be the distance from point B to the center of the large circle. Hence the distance from point B to the center of the small circle is $20-x$.

By similarity, we have that

$$\frac{10}{5} = \frac{x}{20-x}$$
$$2(20-x) = x$$
$$40-2x = x$$
$$3x = 40$$
$$x = \tfrac{40}{3}$$

Therefore, we have that AB $= x - 10 = \frac{40}{3} - 10 = \frac{10}{3}$.

Solution 2: Let x be the length of segment AB. Therefore, the distance from B to the center of the small circle is given by $20-(10+x+5) = 5-x$.

Since the two triangles in the figure are similar, we have that

$$\frac{5}{10} = \frac{(5-x)+5}{x+10} = \frac{10-x}{10+x}$$

Therefore, we have that

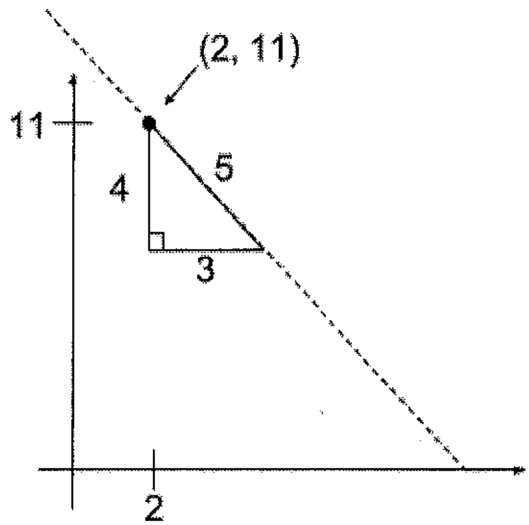
$$5(10+x) = 10(10-x)$$
$$50+5x = 100-10x$$
$$15x = 50$$
$$x = \frac{50}{15} = \frac{10}{3}.$$

3. Find one possible y-intercept for a line $y = \frac{3}{4}x + b$ which is a distance of 6 units from the point (2, 11).

Solution 1 (Similarity): We know that the distance from a point to a line is measured by the length of the shortest possible line segment that connects them. To be as short as possible, we know that the line segment will be perpendicular to the given line.

In this case, since the given line has a slope of $\frac{3}{4}$, we know that the perpendicular line segment will have a slope of $-\frac{4}{3}$.

This line's slope is such that an increase in x by 3 units will lead to a decrease in y by 4 units. We can imagine these changes in x and y as the legs of a right triangle, as seen in the diagram below.



Since the legs are 3 and 4, we know immediately that the hypotenuse is 5. But in this problem, we need a distance of 6 units from the point (2, 11). Therefore, we need a triangle which is similar to the one pictured in the diagram, but which has hypotenuse 6.

By similarity, an increase in the length of the hypotenuse by 20% means we need to increase the length of each leg by 20%. This gives new legs of $1.2 \times 4 = 4.8$ and $1.2 \times 3 = 3.6$.

Therefore, the point which is exactly six units away from the point (2, 11) in the necessary direction is $(2+3.6,\ 11-4.8) = (5.6, 6.2)$

Finally, to compute the possible y intercept, we use the point-slope formula to see that

$$(y-6.2) = \tfrac{3}{4}(x-5.6)$$
$$y = \tfrac{3}{4}x - 4.2 + 6.2$$
$$y = \tfrac{3}{4}x + 2$$

Therefore, one possible y-intercept is 2. Note that if instead we drew the 3-4-5 triangle reflecting a *decrease* in x of 3 and an *increase* in y or 4, we could have used the same logic to find that another possible y-intercept is 17.

Solution 2 (Algebraic): We know that the distance from a point to a line is measured by the length of the shortest possible line segment that connects them. To be as short as possible, we know that the line segment will be perpendicular to the given line.

In this case, since the given line has a slope of $\frac{3}{4}$, we know that the perpendicular line segment will have a slope of $-\frac{4}{3}$. Moreover, since we know the line segment will pass through the point (2,11), we can use the point-slope formula to determine that the perpendicular line segment will be on the line

$$\begin{aligned} (y-11) &= -\tfrac{4}{3}(x-2) \\ y &= -\tfrac{4}{3}x+\tfrac{8}{3}+11 \\ y &= -\tfrac{4}{3}x+\tfrac{41}{3} \end{aligned}$$

To move a distance of six units from the point (2,11) on this line, we need to alter x such that

$$\begin{aligned} 6^2 &= (x-2)^2+(y-11)^2 \\ 36 &= (x-2)^2+((-\tfrac{4}{3}x+13\tfrac{2}{3})-11)^2 \\ 36 &= (x-2)^2+(-\tfrac{4}{3}x+\tfrac{8}{3})^2 \\ 36 &= (x^2-4x+4)+(\tfrac{16}{9}x^2-\tfrac{64}{9}x+\tfrac{64}{9}) \\ 36 &= \tfrac{25}{9}x^2-\tfrac{100}{9}x+\tfrac{100}{9} \\ 324 &= 25x^2-100x+100 \\ 25x^2-100x-224 &= 0 \\ x^2-4x-\tfrac{224}{25} &= 0 \end{aligned}$$

We use the quadratic formula to see

$$x = \frac{4\pm\sqrt{16-4(1)(-\frac{224}{25})}}{2}$$

Since we are only interested in finding one possible intercept, we will only consider the larger possible value for x:

$$x = \frac{4+\sqrt{\frac{1296}{25}})}{2} = 2+\frac{\frac{36}{5}}{2} = 2+\frac{18}{5} = \frac{28}{5}$$

This x value corresponds to a y value of

$$\begin{aligned} y &= -\tfrac{4}{3}(\tfrac{28}{5})+\tfrac{41}{3} \\ y &= -\tfrac{112}{15}+\tfrac{205}{15} = \tfrac{93}{15} = \tfrac{31}{5} \end{aligned}$$

Finally, to compute the desired y-intercept, we use the point-slope formula to get

$$(y-\tfrac{31}{5})=\tfrac{3}{4}(x-\tfrac{28}{5})$$
$$y=\tfrac{3}{4}x-\tfrac{21}{5}+\tfrac{31}{5}$$
$$y=\tfrac{3}{4}x+2$$

Therefore, one possible y-intercept is 2. Note that if instead we considered the smaller possible value of x when solving the quadratic, we could have used the same logic to find that another possible y-intercept is 17.

Team Round

1. If $f(x)$ and $g(x)$ are linear functions such that $f(0)=g(0)=0$. If $f(g(x))=8x$, then evaluate $g(f(10))$.

Solution: Since f and g are linear functions with y-intercepts of 0, we can write

$$f(x)=ax \qquad \text{and} \qquad g(x)=bx$$

Since linear functions are invertible, we have that:

$$f(g(x))=8x \quad \rightarrow \quad g(x)=f^{-1}(8x)$$

Finally we can write that $f^{-1}(x)=-\frac{1}{a}x$. Notice that $f^{-1}(8x)=-\frac{1}{a}8x=8\left(-\frac{1}{a}x\right)=8f^{-1}(x)$. Therefore, we have that

$$g(f(10))=f^{-1}(8f(10))=8f^{-1}(f(10))=8\times 10=80.$$

2. A committee of 5 students is to be chosen at random from 4 juniors and 7 seniors. If the committee is chosen randomly and each student has an equal chance to be chosen, what is the probability that the committee will contain exactly 3 seniors and 2 juniors?

Solution: To determine the probability of this event occurring, we multiply the number of possible ways to choose 3 seniors out of 7 by the number of possible ways to choose 2 juniors out of 4, and ultimately divide by the number of ways to choose 5 people out of a group of 11. This is computed as

$$\frac{(7\ choose\ 3)\times(4\ choose\ 2)}{(11\ choose\ 5)} = \frac{\binom{7}{3}\binom{4}{2}}{\binom{11}{5}} = \frac{{}_7C_3 \times {}_4C_2}{{}_{11}C_5} = \frac{\frac{7!}{3!4!}\times\frac{4!}{2!2!}}{\frac{11!}{5!6!}} = \frac{\frac{7\times6\times5\times4}{2\times1\times2\times1}}{\frac{11\times10\times9\times8\times7}{5\times4\times3\times2\times1}} =$$

$$= 7\times6\times5\ \frac{5\times4\times3\times2\times1}{11\times10\times9\times8\times7} = \frac{5}{11}$$

3. If $f(x) = \frac{\sqrt{x}}{2+\sqrt{x}}$ and $g(x) = \left(\frac{2x}{1-x}\right)^2$, then compute $g(f(x))$ and simplify.

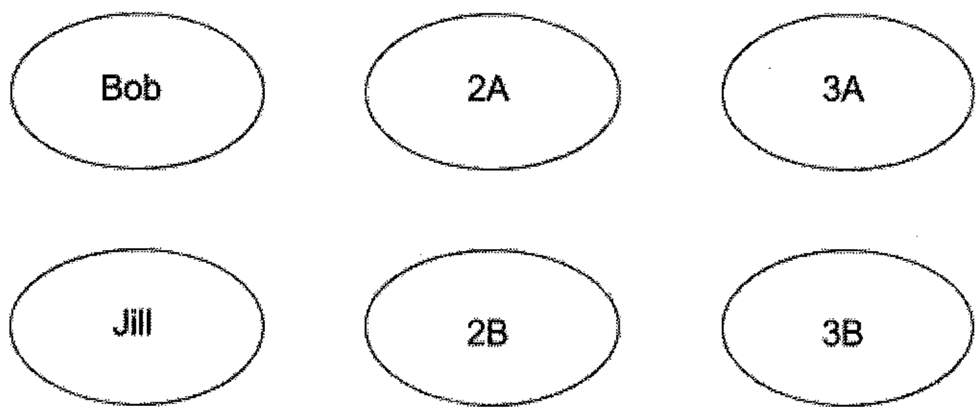
Solution: We have that

$$g(f(x)) = \left(\frac{2f(x)}{1-f(x)}\right)^2 = \left(\frac{2\left(\frac{\sqrt{x}}{2+\sqrt{x}}\right)}{1-\frac{\sqrt{x}}{2+\sqrt{x}}}\right)^2 = \left(\frac{2\left(\frac{\sqrt{x}}{2+\sqrt{x}}\right)}{1-\frac{\sqrt{x}}{2+\sqrt{x}}}\cdot\frac{2+\sqrt{x}}{2+\sqrt{x}}\right)^2 =$$

$$= \left(\frac{2\sqrt{x}}{2+\sqrt{x}-\sqrt{x}}\right)^2 = (\sqrt{x})^2 = x$$

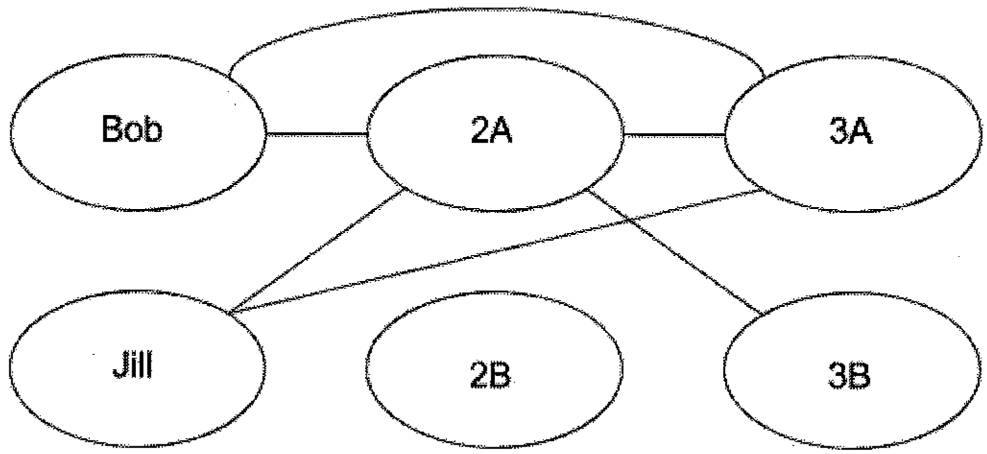
4. Bob and Jill meet 2 other married couples at a restaurant. No person shakes hand with his or her spouse. Excluding Bob, each other person shakes hands with a different number of people (possibly with no one) and does not shake hands with the same person twice. With how many people did Bob shake hands?

Solution: Draw a diagram of the situation, where 2A and 2B are one couple and 3A and 3B are the other couple:



If we represent handshakes lines connecting two ovals, we know that there will be no vertical lines, since no one shakes hands with their spouse. Moreover, we know that outside of Bob, no two people shake hands with the same number of people and no one shakes hands twice with the same person.

Since there are a total of 6 people and no shakes occur between spouses, the most number of handshakes that one person can be part of is 4. Therefore, we need to assign 0 through 4 handshakes uniquely to Jill, 2A, 2B, 3A and 3B. One solution is as follows:



Person 2A shakes hands with 4 people, 3A shakes 3 times, Jill shakes 2 times, 3B shakes once and 2B makes no handshakes. Other feasible drawings can be made, but Bob will always shake hands twice.

5. Find the x coordinate of the focus of the conic section given by $y^2+4y-6x+22=0$.

Solution: Begin by noting that the curve described is a horizontal parabola opening to the right - to see this we can rearrange the equation as $x=\frac{1}{6}(y^2+4y+22)$. Now we will compute the vertex of the parabola.

Split up the constant in order to complete the square:

$$y^2+4y-6x+22=0$$
$$y^2+4y-6x+(4+18)=0$$
$$(y^2+4y+4)-6x+18=0$$
$$(y+2)^2=6(x-3)$$

The equation is now in vertex form, giving us that the parabola's vertex is at (3, -2).

For a horizontal parabola, the distance a between its vertex and focus is equal to $\frac{1}{4}$ of the coefficient on the x term in the vertex form of the equation. In this case, we have that the coefficient on the x term is 6, meaning that

$$a=\frac{1}{4}\times 6 \rightarrow a=\frac{3}{2}$$

Since a is positive, we know that the focus occurs to the right of the vertex (which makes sense given that we already determined that this parabola opens to right). Since the vertex of the parabola is at (3,-2), we have that the focus is located at $(3+\frac{3}{2},-2)$. This gives that the x-coordinate of the focus is $\frac{9}{2}$.

6. Solve and graph on the real number line: $x(x+2)(x-3)>0$

Solution 1: The expression multiplies three factors:

(1) x

(2) $(x+2)$

(3) $(x-3)$

There are a total of four ways that the inequality can hold.

1. (1), (2) and (3) all positive: $\{x>0\}\cap\{x>-2\}\cap\{x>-3\}$ $\Rightarrow$ $x>3$
2. (1) and (2) negative, (3) positive: $\{x<0\}\cap\{x<-2\}\cap\{x>3\}$ $\Rightarrow$ $\varnothing$
3. (1) and (3) negative, (2) positive: $\{x<0\}\cap\{x>-2\}\cap\{x<3\}$ $\Rightarrow$ $-2<x<0$
4. (1) positive, (2) and (3) negative: $\{x>0\}\cap\{x<-2\}\cap\{x<3\}$ $\Rightarrow$ $\varnothing$

The union of all of these sets yields $\{x : x>3 \text{ or } -2<x<0\}$

Solution 2: Since the expression has three factors x, $(x+2)$ and $(x-3)$, we know it has roots at $x=0$, $x=-2$ and $x=3$. Since polynomials are continuous, we know that at all points between roots the polynomial will be either positive or negative collectively. The same is true for all points less than the smallest root and greater than the largest root. Therefore, to answer to question, we just need to test a single point in each of the intervals that we have.

Pick a value greater than 3 such as 5: each factor is positive, so the product is greater than 0. This means the interval $\{x : x>3\}$ is part of the solution.

Pick a value between 0 and 3 such as 1: the factors are 1, 3, and -2 so the product is negative and does not satisfy the equation. Hence, $\{x : 0<x<3\}$ is <u>not</u> part of the solution.

Pick a value between -2 and 0 such as -1: the factors are -1, 1, and -4, so the product will be positive. Hence, the interval $\{x : -2<x<0\}$ is part of the solution.

Pick a value less than -2 such as -5: each factor is negative, so the product will be negative and not satisfy the equation. Hence $\{x : x< -2\}$ is <u>not</u> part of the solution.

Taking the union of all the successful intervals gives that the solution is $\{x : x>3 \text{ or } -2<x<0\}$.

7. Let x be the largest integer such that 2^x divides 100! evenly. Compute x.

Solution 1: We know that 100! is the product of the integers 1 through 100. To answer the question, we need to determine how many factors of 2 occur in each of the integers 1 to 100.

Integers w/ only 1 factor of 2: 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98
(25 integers in this list → 25 factors of 2)

The integers which have 2 factors of 2 can be found by multiplying the above list by 2 and removing any number larger than 100.

Integers w/ 2 factors of 2: 4, 12, 20, 28, 36, 44, 52, 60, 68, 76, 84, 92, 100
(13 integers in this list → $13 \times 2 = 26$ factors of 2)

Likewise, we can find the integers with 3 factors of 3 by multiplying each number in the 2 factors of 2 list by 2 and removing any number larger than 100 (etc.).

Integers w/ 3 factors of 2: 8, 24, 40, 56, 72, 88
(6 integers in this list → $6 \times 3 = 18$ factors of 2)

Integers w/ 4 factors of 2: 16, 48, 80
(3 integers in this list → $3 \times 4 = 12$ factors of 2)

Integers w/ 5 factors of 2: 32, 96
(2 integers in this list → $2 \times 5 = 10$ factors of 2)

Integers w/ 6 factors of 2: 64
(1 integer in this list → $1 \times 6 = 6$ factors of 2)

Adding up all the factors of 2 we have $6 + 10 + 12 + 18 + 26 + 25 = 97$.

Solution 2: We know that 100! represents the product of the numbers from 1 to 100. The goal in this problem is to count the total factors of 2 in these integers. Notice that:

Every other integer will be divisible by 2 so there are 50 such integers.
Every 4th integer will be divisible by 4 so there are 25 such integers.
Every 8th integer will be divisible by 8 so there are 12 such integers.
Every 16th integer will be divisible by 16 so there are 6 such integers.
Every 32nd integer will be divisible by 32 so there are 3 such integers.
Every 64th integer will be divisible by 64 so there is 1 such integer.

The total number of factors of 2 then is $50 + 25 + 12 + 6 + 3 + 1 = 97$, meaning that x must be equal to 97.

8. Two positive numbers a and b have the property that their sum is equal to their product. If $a - b = 3$, then determine the value of $\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}}$.

Solution: We are given two equations:

$$a + b = ab \qquad (1)$$

$$a - b = 3 \qquad (2)$$

Substitute equation (2) into equation (1) to get

$$(b+3) + b = (b+3)b$$

$$2b + 3 = b^2 + 3b$$

$$b^2 + b - 3 = 0$$

Use the fact that we know that b is positive to get

$$b = \frac{-1+\sqrt{1-4(1)(-3)}}{2} = \frac{\sqrt{13}-1}{2}.$$

And so we have that

$$a = b + 3 = \frac{\sqrt{13}+5}{2} \quad \text{which gives } a+b = \frac{\sqrt{13}+5}{2} + \frac{\sqrt{13}-1}{2} = \sqrt{13}+2.$$

Now simplify the original expression making use of equation (1):

$$\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}} = \frac{1}{\frac{a^2+b^2}{a^2b^2}} = \frac{a^2b^2}{a^2+b^2} = \frac{(ab)^2}{(a+b)^2-2ab} = \frac{(a+b)^2}{(a+b)^2-2(a+b)} = \frac{a+b}{a+b-2}.$$

Now we can plug in the value of $a+b$ we computed to get that

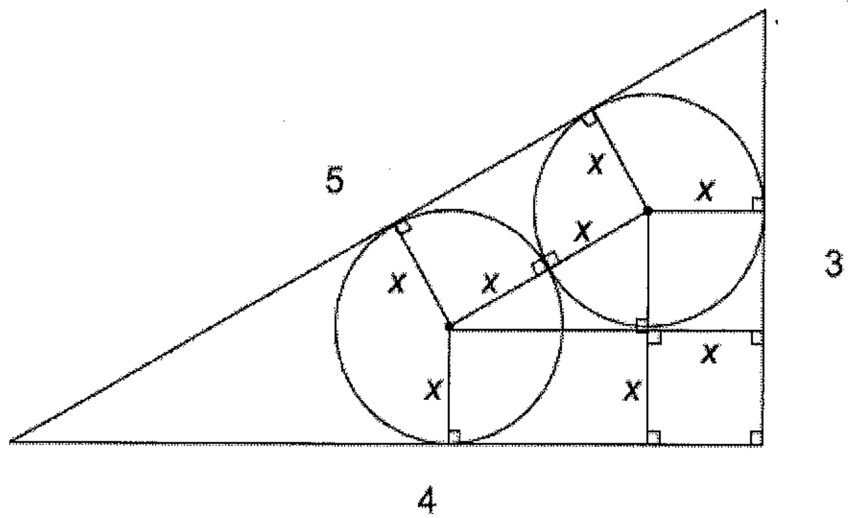
$$\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}} = \frac{a+b}{a+b-2} = \frac{\sqrt{13}+2}{\sqrt{13}+2-2} = \frac{\sqrt{13}+2}{\sqrt{13}}.$$

Rationalizing the expression gives

$$\frac{\sqrt{13}+2}{\sqrt{13}} = \frac{13+2\sqrt{13}}{13}.$$

9. The lengths of the legs of a right triangle are 3 and 4. Two congruent circles, externally tangent to each other, are drawn inside the triangle such that each circle is tangent to the hypotenuse and one of the legs. Determine the distance between the circles' centers.

Solution: Let x be the radius of the two circles. We can make the following drawing, noting that since the legs are 3 and 4, the hypotenuse must be 5:



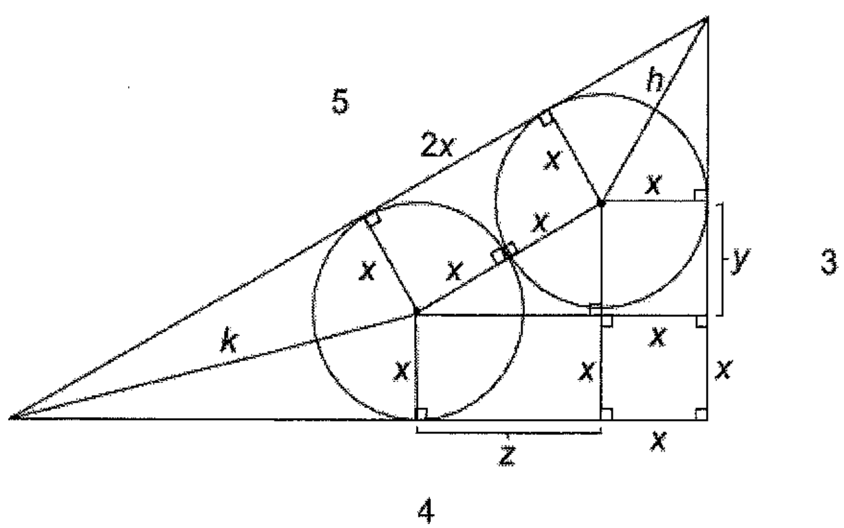
Notice that the small triangle is similar to the large triangle. The hypotenuse of the small triangle is $2x$. Let y be the small triangle's smaller leg and z be the small triangle's larger leg. By similarity, we have that there exists some k such that

$$2x = 5k \quad (1)$$

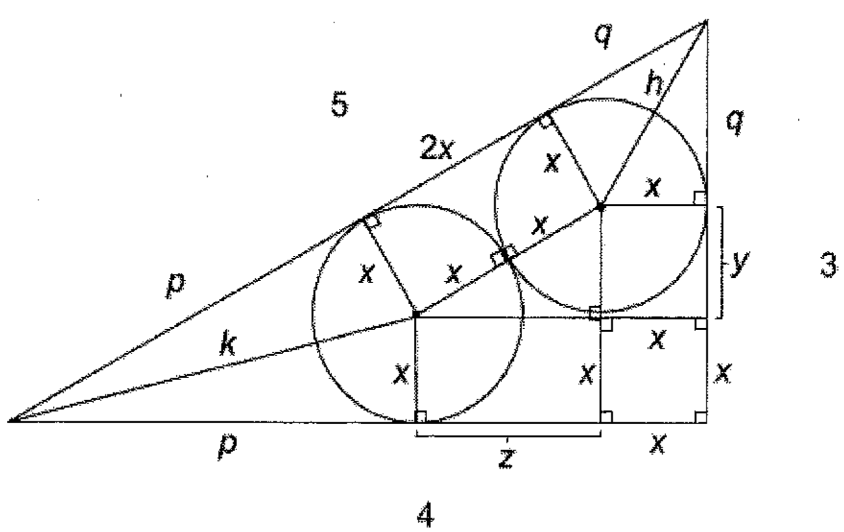
$$y = 3k \quad (2)$$

$$z = 4k \quad (3)$$

We now have four variables but only three equations. To solve we need more information. Fill in the drawing a bit more, drawing lines from the corners of the large triangle to the centers of the circles.



Let h and k be the lengths of these newly drawn segments and note that each segment has now created two new small right triangles. Begin by examining the two triangles which have hypotenuse k and leg x. Since these are both right triangles, the Pythagorean Theorem guarantees that the second leg in each of these triangles must be equal to one another; we will call this distance p. The same logic can be applied to the triangles with hypotenuse h and leg x. Let q be the length of the second legs of each of these triangles.



With these new lengths, we now have that

$$2x + p + q = 5 \qquad (4)$$

$$x + y + p = 3 \qquad (5)$$

$$x + z + q = 4 \qquad (6)$$

Plugging equations (1), (2) and (3) into equations (4), (5) and (6) gives us

$$5k + p + q = 5 \qquad (4^*)$$

$$5.5k + p = 3 \qquad (5^*)$$

$$6.5k + q = 4 \qquad (6^*)$$

Subtract (5*) and (6*) from (4*) to get

$$5k - 12k + p + q - p - q = 5 - 7$$

$$-7k = -2$$

$$k = \tfrac{2}{7}$$

We are interested in the distance between the centers of the two circles, which is precisely equal to $2x = 2(2.5k) = 5k = 5 \times \frac{2}{7} = \frac{10}{7}$.